MIDTERM I

Université d'Ottawa • University of Ottawa

MAT 1320 D

WEDNESDAY, FEBRUARY 4, 2009 **Name:** Solutions

Student Number: ____________________

Read all of the following information before starting the exam:

- Only basic scientific calculators (non-programmable, non-graphing, no differentiation or integration capability) are allowed on the exam.
- Notebooks, notes, cheating sheets, and books are NOT permitted.
- Students must present their student cards if asked.
- Students may not leave until one hour after the examination has begun.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- **Circle or otherwise indicate your final answers.**
- This test has **SIX** problems and is worth **100** points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Question	1	2	3	4	5	6	TOTAL
Points	15	15	15	15	20	20	100
Mark							

1. (*15 points*) If $f(x) = \ln x$ and $g(x) = x^2 - 9$, find the functions $f \circ g$, $g \circ f$, $f \circ f$, and their domains.

(sol) $f(x) = \ln x \quad : D = (0, \infty)$

$g(x) = x^2 - 9 \quad : D = \mathbb{R}$

■ $(f \circ g)(x) = f(g(x)) = f(x^2 - 9) = \underline{\ln(x^2 - 9)}$

⇒ **Domain**

$x^2 - 9 > 0 \Rightarrow (x-3)(x+3) > 0$

$\therefore \underline{\{x \mid x > 3,\ x < -3\} \text{ or } (-\infty, -3) \cup (3, \infty)}$

■ $(g \circ f)(x) = g(f(x)) = g(\ln x) = \underline{(\ln x)^2 - 9}$

⇒ **Domain**

$x > 0$

$\therefore \underline{\{x \mid x > 0\} \text{ or } (0, \infty)}$

■ $(f \circ f)(x) = f(f(x)) = f(\ln x) = \underline{\ln(\ln x)}$

⇒ **Domain**

$\ln x > 0 \Rightarrow e^{\ln x} > e^0 \Rightarrow x > 1$

$\therefore \underline{\{x \mid x > 1\} \text{ or } (1, \infty)}$

2. (*15 points*) Let f such that

$$f(x) = \begin{cases} x - m, & \text{if } x < 3, \\ 1 - mx, & \text{if } x \geq 3. \end{cases}$$

Find m so that f is continuous for all x.

(The function $f(x)$ is continuous for $x<3$ and $x \geq 3$ because each piece of f is a polynomial.)

(sol) For f to be continuous at $x=3$, the two one-sided limits must exist and be equal.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(x-m) = 3-m$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(1-mx) = 1-3m$$

$\Rightarrow \underbrace{3-m = 1-3m}$

$\Rightarrow 2m = -2$

$\Rightarrow \boxed{m=-1}$ //

<Note>

For $m=-1$,

$$\lim_{x\to 3^-} f(x) = 3-(-1) = 4$$

$$\lim_{x\to 3^+} f(x) = 1-3(-1) = 4$$

$\Rightarrow \lim_{x\to 3} f(x) = \underline{4}.$

Also, $f(3) = 1 - m(3) = 1-(-1)\cdot 3 = \underline{4}$

($m=-1$)

Therefore, $\lim_{x\to 3} f(x) = 4 = f(3)$.

■

3. (*15 points*) Find $\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right)$.

(sol) $\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right)\cdot\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$

$= \lim_{x\to\infty} \dfrac{\cancel{x^2}+x - \cancel{x^2}}{\sqrt{x^2+x}+x}$

$= \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+x}+x}$

$= \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x}$

$= \lim_{x\to\infty} \dfrac{x}{\sqrt{x^2}\sqrt{1+\frac{1}{x}}+x} = \lim_{x\to\infty} \dfrac{x}{|x|\sqrt{1+\frac{1}{x}}+x}$ $\left(|x|=x \because x>0 \text{ as } x\to\infty\right)$

$= \lim_{x\to\infty} \dfrac{x}{x\sqrt{1+\frac{1}{x}}+x}$ (Divide top & bottom by 'x'.)

$= \lim_{x\to\infty} \dfrac{\frac{x}{x}}{\frac{x\sqrt{1+\frac{1}{x}}+x}{x}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{x}}+1} = \boxed{\frac{1}{2}}.$

4. (*15 points*) Find the inverse function of $f(x) = \dfrac{4x-1}{2x+3}$. What is the domain of the inverse function?

(sol) $y = \dfrac{4x-1}{2x+3}$

$\Rightarrow y(2x+3) = 4x-1$

$\Rightarrow 2xy + 3y = 4x-1$

$\Rightarrow 2x(y-2) = -(3y+1)$

$\Rightarrow x = \dfrac{-(3y+1)}{2(y-2)}$

$\therefore f^{-1}(x) = \dfrac{3x+1}{2(2-x)}$ or $\dfrac{3x+1}{4-2x}$

* Domain of $f^{-1} = \{x \mid x \neq 2\}$.

5. (*20 points*) Let $f(x) = x^3 - 4x^2 + 9$.

a. (*15 pts*) Using the definition of the derivative, find the slope of the tangent line at the point $(a, f(a))$.

b. (*5 pts*) What is the equation of the tangent line at the point $(2, 1)$?

(sol) (a)

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{(a+\Delta x)^3 - 4(a+\Delta x)^2 + 9 - (a^3 - 4a^2 + 9)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{a^3} + 3a^2\Delta x + 3a(\Delta x)^2 + (\Delta x)^3 - 4[\cancel{a^2} + 2a(\Delta x) + (\Delta x)^2] + \cancel{9} - \cancel{a^3} + \cancel{4a^2} - \cancel{9}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}[3a^2 + 3a(\Delta x) + (\Delta x)^2 - 8a - 4(\Delta x)]}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} [3a^2 + 3a(\Delta x) + (\Delta x)^2 - 8a - 4(\Delta x)]$$

$$= \underline{3a^2 - 8a}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

$$\lim_{x \to a} \frac{x^3 - 4x^2 + 9 - (a^3 - 4a^2 + 9)}{x - a}$$

$$= \lim_{x \to a} \frac{(x^3 - a^3) - 4(x^2 - a^2)}{x - a}$$

$$= \lim_{x \to a} \frac{(x-a)(x^2 + ax + a^2) - 4(x-a)(x+a)}{(x-a)}$$

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^2 + ax + a^2 - 4x - 4a)}{\cancel{(x-a)}}$$

$$= \lim_{x \to a} (x^2 + ax + a^2 - 4x - 4a)$$

$$= \underline{3a^2 - 8a}$$

Thus, the slope of the tangent line at $(a, f(a))$ is $\boxed{3a^2 - 8a}$ //

(b).

At $\underbrace{(2,1)}_{(a,\, f(a))}$, $m = 3a^2 - 8a \Big]_{a=2} = 3 \cdot (2)^2 - 8 \cdot 2 = 12 - 16 = \underline{-4}$

$\therefore\ y - 1 = -4(x-2)$

$\Rightarrow\ y = -4x + 8 + 1$

Therefore, $\boxed{y = -4x + 9}$.

6. (*20 points*) The half-life of palladium-100, ^{100}Pd, is four days. (So half of any given quantity of ^{100}Pd will disintegrate in four days.) The initial mass of a sample is one gram.

a. (*10 pts*) Find the mass $m(t)$ that remains after t days.

b. (*5 pts*) Find the mass that remains after 16 days.

c. (*5 pts*) When will the mass be reduced to 0.01 g?

(sol). (a).

$$m(0) = 1 \, (g)$$

$$m(4) = \frac{1}{2} \cdot 1 = \frac{1}{2} \, (g)$$

$$m(8) = \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^2 \, (g).$$

$$\vdots$$

$$\underline{m(t) = \left(\frac{1}{2}\right)^{t/4} = 2^{-t/4}} \quad //$$

(b) $m(16) = 2^{-16/4} = 2^{-4} = \underline{\underline{\frac{1}{16}\ g}}$ //

(c). $0.01 = 2^{-t/4}$

$$\Rightarrow \ln 0.01 = \ln 2^{-t/4}$$

$$\Rightarrow \ln 0.01 = -\frac{t}{4} \ln 2$$

$$\Rightarrow -\frac{t}{4} = \frac{\ln 0.01}{\ln 2}$$

$$\therefore \underline{t = -4\left(\frac{\ln 0.01}{\ln 2}\right) \approx 26.6 \text{ days.}} \quad //$$